Divergence Theorem and Its Application in Characterizing Fluid Flow

Let $\mathbf{v}$ be the velocity of flow of a fluid element and $\rho(x, y, z, t)$ be the mass density of fluid at a point (x, y, z) at time t. Thus, $\mathbf{q} = \rho\mathbf{v}$ represents a vector in the direction of flow with magnitude equal to the mass flow rate per unit area. Then

$$\mathbf{q} \bullet d\vec{\sigma} = (\mathbf{q} \bullet \mathbf{n})dA \tag{1}$$

represents the differential mass flow rate through a directed element of surface area, i.e. $d\vec{\sigma} = \mathbf{n}dA$.

Let

$$\mathbf{q} = q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k} \tag{2}$$

Let us consider an *closed* differential volume element defined by

$$\bar{x} \leq x \leq \bar{x} + dx$$

$$\bar{y} \leq y \leq \bar{y} + dy$$

$$\bar{z} \leq z \leq \bar{z} + dz$$

The surface vector at $y = \bar{y}$ can be expressed as

$$-\mathbf{j}dxdz$$

Notice that the minus sign is used to denote the outward direction. Thus, the differential mass flow rate outward though this surface is

$$\mathbf{q} \bullet [-\mathbf{j}(dx)(dz)] = -q_y dxdz \tag{3}$$

Therefore, if q_y in this equation is positive, the flow through this surface is *into* the volume element.

On the other hand, the differential flow rate through the surface at $y = \bar{y} + dy$ is

$$(dxdz\mathbf{j}) \bullet \mathbf{q}(\bar{x}, \bar{y} + dy, \bar{z}) \cong \left(q_y + \frac{\partial q_y}{\partial y} dy \right) dxdz \tag{4}$$

Note that the inner product on the left hand side of the above equation results in $q_y(\bar{x}, \bar{y} + dy, \bar{z})dxdz$. Notice also that similar treatment can be carried out for the other 4 faces.

The *net* differential *outward* flow rate is

$$\begin{aligned} dF &= \left[\left(q_x + \frac{\partial q_x}{\partial x} dx \right) - q_x \right] dydz \\ &+ \left[\left(q_y + \frac{\partial q_y}{\partial y} dy \right) - q_y \right] dxdz \\ &+ \left[\left(q_z + \frac{\partial q_z}{\partial z} dz \right) - q_z \right] dxdy \\ &= \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) dxdydz \end{aligned}$$

Thus

$$dF = (\nabla \bullet \mathbf{q})\, dV = (\text{ div } \mathbf{q})\, dV \tag{5}$$

This result implies that $\nabla \bullet \mathbf{q}$ at a point P represents the rate of outward fluid flow per unit volume across the boundary from a differential volume associated with P.

If no mass is added (generated) or withdrawn (disappeared) within the boundary of the differential volume dV, the mass balance can be written as

$$-\frac{\partial \rho}{\partial t} dV = dF = (\nabla \bullet \mathbf{q})\, dV \tag{6}$$

or

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{v}) = 0 \tag{7}$$

This is the so-called *equation of continuity*.

For incompressible fluid, since the density is constant, the above equation can be written as

$$\nabla \bullet \mathbf{v} = 0 \tag{8}$$

It is also clear that for incompressible fluid

$$\rho \nabla \bullet \mathbf{v} = \nabla \bullet \mathbf{q} = 0 \tag{9}$$

Thus, the net outward flow rate across the boundary of the differential volume element is zero, i.e.

$$dF = (\nabla \bullet \mathbf{q})\, dV = 0 \tag{10}$$

If there is a point where mass is added to (or generated in) the differential volume within the boundary, then

$$\frac{\partial \rho}{\partial t} dV = -\,(\nabla \bullet \mathbf{q})\, dV + \tilde{S} dV \tag{11}$$

where, $\tilde{S}$ is the rate of addition per unit volume. For incompressible fluid, $\partial \rho / \partial t = 0$. Thus,

$$\tilde{S} = \nabla \bullet \mathbf{q} = \rho \nabla \bullet \mathbf{v} \tag{12}$$

We speak of points where fluid is added to or taken from the system as *sources* and *sinks* respectively. From equation (12), we can see that a source is associated with positive $\nabla \bullet \mathbf{v}$.

Now, consider a closed bounded region $\mathcal{V}$ of incompressible fluid in 3-dimensional space. Suppose there are sources in each differential volume dV in the region. Thus, for each differential volume

$$\tilde{S} dV = dF = \rho (\nabla \bullet \mathbf{v}) dV \tag{13}$$

is the amount of fluid introduced in dV per unit time (i.e. LHS) or the net outward fluid flow rate across the boundary of the differential volume (i.e. RHS). Thus, the net generation rate of fluid mass within the region $\mathcal{V}$ through sources and/or sinks is

$$\int\int\int_{\mathcal{V}} \tilde{S} dV = \rho \int\int\int_{\mathcal{V}} (\nabla \bullet \mathbf{v}) dV \tag{14}$$

If the total mass is conserved and the fluid is incompressible, this fluid clearly must escape from the region $\mathcal{V}$ through the surface $\mathcal{S}$ which bounds it. Let us use $d\vec{\sigma}$ to represent a surface element vector and

$$d\vec{\sigma} = \mathbf{n} dA \tag{15}$$

Thus, the outward fluid flow through the surface element is

$$\rho \mathbf{v} \bullet d\vec{\sigma} = \rho \mathbf{v} \bullet \mathbf{n} dA \tag{16}$$

The total outward flow rate through $\mathcal{S}$ is

$$\rho \oint\oint_{\mathcal{S}} \mathbf{v} \bullet d\vec{\sigma} = \rho \oint\oint_{\mathcal{S}} (\mathbf{v} \bullet \mathbf{n}) dA \tag{17}$$

where, the symbol "$\oint\oint_{\mathcal{S}}$" denotes the integration over closed surface $\mathcal{S}$. Equating the right hand sides of equations (14) and (17), one obtain the Divergence Theorem, i.e.

$$\int\int\int_{\mathcal{V}} (\nabla \bullet \mathbf{v}) dV = \oint\oint_{\mathcal{S}} (\mathbf{v} \bullet \mathbf{n}) dA \tag{18}$$

This theorem can be proved without referring to the physical consideration. It can be directly established mathematically. As long as $\mathbf{v}$ and its partial derivatives are continuous in $\mathcal{V}$ and on $\mathcal{S}$ and if $\mathcal{S}$ is piecewise continuous, the Divergence Theorem is always valid.

In a space region $\mathcal{V}$,

1. $\nabla \times \mathbf{F} = 0$ (everywhere) implies that

$$\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = 0$$

   In other words, the net circulation of **F** around a closed curve $\mathcal{C}$ in $\mathcal{V}$ is zero.

2. $\nabla \bullet \mathbf{F} = 0$ (everywhere) implies that

$$\oint\oint_{\mathcal{S}} \mathbf{F} \bullet d\vec{\sigma} = 0$$

   In other words, the net flux of **F** through a closed surface $\mathcal{S}$ in $\mathcal{V}$ is zero.

Green's Theorem and Its Application in Characterizing Heat Conduction

Let $\mathbf{v}_1 = \varphi_1 \nabla \varphi_2$ and substitute into the Divergence Theorem:

$$\int\int\int_{\mathcal{V}} \nabla \bullet \varphi_1 \nabla \varphi_2 dV = \oint\oint_{\mathcal{S}} \mathbf{n} \bullet \varphi_1 \nabla \varphi_2 dA \tag{19}$$

By making use of the identity

$$\nabla \bullet \phi \mathbf{u} = \phi \nabla \bullet \mathbf{u} + \mathbf{u} \bullet \nabla \phi$$

the following *First Form of Green's Theorem* can be obtained:

$$\int\int\int_{\mathcal{V}} \left(\varphi_1 \nabla^2 \varphi_2 + \nabla \varphi_1 \bullet \nabla \varphi_2 \right) dV = \oint\oint_{\mathcal{S}} \left(\mathbf{n} \bullet \varphi_1 \nabla \varphi_2 \right) dA \tag{20}$$

Let $\mathbf{v}_2 = \varphi_2 \nabla \varphi_1$ and substitute into the Divergence Theorem again. Subtracting the resulting equation from equation (20), one can derive the *Second Form of Green's Theorem*, i.e.,

$$\int\int\int_{\mathcal{V}} \left(\varphi_1 \nabla^2 \varphi_2 - \varphi_2 \nabla^2 \varphi_1 \right) dV = \oint\oint_{\mathcal{S}} \mathbf{n} \bullet \left(\varphi_1 \nabla \varphi_2 - \varphi_2 \nabla \varphi_1 \right) dA \tag{21}$$

The following special cases of the Green's Theorem can also be derived form the two original forms:

1. $\varphi_1 = \varphi_2 = \varphi$

   The first form becomes

$$\int\int\int_{\mathcal{V}} \left[\varphi \nabla^2 \varphi + (\nabla \varphi)^2\right] dV = \oint\oint_{\mathcal{S}} \varphi \left(\mathbf{n} \bullet \nabla \varphi\right) dA \qquad (22)$$

   Notice that $\mathbf{n} \bullet \nabla \varphi$ represents the derivative of φ in the direction of $\mathbf{n}$ and $\mathbf{n}$ is the outward normal vector at the given point on $\mathcal{S}$, i.e.

$$\frac{\partial \varphi}{\partial n} \equiv \mathbf{n} \bullet \nabla \varphi$$

   Thus,

$$\int\int\int_{\mathcal{V}} \left[\varphi \nabla^2 \varphi + (\nabla \varphi)^2\right] dV = \oint\oint_{\mathcal{S}} \left(\varphi \frac{\partial \varphi}{\partial n}\right) dA \qquad (23)$$

2. $\varphi_1 = \varphi$ and $\varphi_2 = 1$

   The second form becomes

$$\int\int\int_{\mathcal{V}} \left(\nabla^2 \varphi\right) dV = \oint\oint_{\mathcal{S}} \left(\mathbf{n} \bullet \nabla \varphi\right) dA = \oint\oint_{\mathcal{S}} \left(\frac{\partial \varphi}{\partial n}\right) dA \qquad (24)$$

Let us consider the heat flow in a region in space such that the temperature $T = T(x, y, z, t)$. For any region $\mathcal{V}$ bounded by a closed surface $\mathcal{S}$, the rate at which heat is absorbed by a volume element dV is

$$dQ_1 = \rho C_p \frac{\partial T}{\partial t} dV \qquad (25)$$

If there is no sources or sinks in $\mathcal{V}$, the net rate of heat flow into $\mathcal{V}$ is

$$Q_1 = \int\int\int_{\mathcal{V}} \left(\rho C_p \frac{\partial T}{\partial t}\right) dV \qquad (26)$$

Consider the outward heat flow rate from $\mathcal{V}$ across a surface element on $\mathcal{S}$, i.e.

$$-dQ_2 = -k \frac{\partial T}{\partial n} dA = -k \left(\nabla T \bullet \mathbf{n}\right) dA \qquad (27)$$

Thus, the net heat flow rate into $\mathcal{V}$ is

$$Q_2 = \oint \oint_{\mathcal{S}} k \left(\nabla T \bullet \mathbf{n} \right) dA \tag{28}$$

Since $Q_1 = Q_2$, one can obtain

$$\int \int \int_{\mathcal{V}} \left(\rho C_p \frac{\partial T}{\partial t} \right) dV = \oint \oint_{\mathcal{S}} k \left(\nabla T \bullet \mathbf{n} \right) dA \tag{29}$$

According to the special case 2 of Green's Theorem, the RHS of the above equation can be substituted by

$$\oint \oint_{\mathcal{S}} k \left(\nabla T \bullet \mathbf{n} \right) dA = \int \int \int_{\mathcal{V}} k \nabla^2 T dV \tag{30}$$

Thus,

$$\int \int \int_{\mathcal{V}} \left(\rho C_p \frac{\partial T}{\partial t} - k \nabla^2 T \right) dV = 0 \tag{31}$$

$$\Rightarrow \qquad \frac{\partial T}{\partial t} = \alpha \nabla^2 T \tag{32}$$

where,

$$\alpha = \frac{k}{\rho C_p}$$

At steady state, $\partial T / \partial t = 0$. Thus,

$$\nabla^2 T = 0 \tag{33}$$

This is the so-called *Laplace equation.* Let us consider two types of boundary conditions:

- (i) T is prescribed on $\mathcal{S}$;
- (ii) The outward heat flow rate $-k \frac{\partial T}{\partial n}$ is prescribed on $\mathcal{S}$.

The uniqueness of solution to the Laplace equation can be determined with Green's Theorem. The analysis is given in the sequel:

Case (i): Dirichlet Problem

From special case 1 of the Green's Theorem, one can obtain

$$\int\int\int_{\mathcal{V}} \left[T\nabla^2 T + (\nabla T)^2\right] dV = \oint\oint_{\mathcal{S}} \left(T\frac{\partial T}{\partial n}\right) dA \tag{34}$$

By substituting the Laplace equation, we can obtain

$$\int\int\int_{\mathcal{V}} (\nabla T)^2 dV = \oint\oint_{\mathcal{S}} \left(T\frac{\partial T}{\partial n}\right) dA \tag{35}$$

Let us next assume two solutions exist, i.e. $\nabla^2 T_1 = 0$ and $\nabla^2 T_2 = 0$, then $\nabla^2(T_1 - T_2) = 0$. Thus, $T_1 - T_2$ is also a solution. Substituting $(T_2 - T_1)$ into equation (35) yields the following result:

$$\int\int\int_{\mathcal{V}} \left[\nabla(T_2 - T_1)\right]^2 dV = \oint\oint_{\mathcal{S}} \left[(T_2 - T_1)\frac{\partial(T_2 - T_1)}{\partial n}\right] dA = 0 \tag{36}$$

This is due to the fact that temperature is prescribed on $\mathcal{S}$. As a result,

$$\nabla(T_2 - T_1) = 0 \tag{37}$$

Thus, $T_2 - T_1$ is constant in $\mathcal{V}$. However,since $T_1 = T_2$ on $\mathcal{S} \subset \mathcal{V}$, one can conclude that $T_1 = T_2$ in $\mathcal{V}$, i.e., the solution of Laplace equation is unique if temperature is prescribed on boundary surface.

Case (ii): Neumann Problem

The 2nd special case of Green's Theorem and Laplace equation can be used to produce the following equation:

$$\int\int\int_{\mathcal{V}} \left(\nabla^2 T\right) dV = \oint\oint_{\mathcal{S}} \left(\frac{\partial T}{\partial n}\right) dA = 0 \tag{38}$$

Thus, we can conclude that the net flow across boundary $\mathcal{S}$ must be zero. This is obvious for *steady state* heat flow without sources and/or sinks. In other words, $\partial T/\partial n$ cannot be prescribed arbitrarily. Its mean value on $\mathcal{S}$ must be zero.

Let us again assume that T_1 and T_2 are two distinct solutions of the Laplace equation. Then it can be shown as before that $T_2 - T_1$ is also a solution. Substituting $T_2 - T_1$ into equation (35) yields

$$\int\int\int_{\mathcal{V}} [\nabla(T_2 - T_1)]^2 \, dV = \oint\oint_{\mathcal{S}} \left[(T_2 - T_1)\frac{\partial(T_2 - T_1)}{\partial n}\right] dA = 0 \tag{39}$$

This is due to the fact that the outward heat flow rate $-k\frac{\partial T}{\partial n}$ is prescribed on $\mathcal{S}$, i.e.

$$\frac{\partial T_1}{\partial n} = \frac{\partial T_2}{\partial n}$$

However, an arbitrary constant C must be included in the general solution in this case, i.e.

$$T_2 - T_1 = C \tag{40}$$

Circulation

Suppose that the motion of a fluid element is simply a rotation about a given axis in space. Thus,

$$\mathbf{v} = \vec{\omega} \times \mathbf{r} \tag{41}$$

$$\begin{aligned} \nabla \bullet \mathbf{v} &= \text{div } \mathbf{v} = \nabla \bullet (\vec{\omega} \times \mathbf{r}) \\ &= \mathbf{r} \bullet (\nabla \times \vec{\omega}) - \vec{\omega} \bullet (\nabla \times \mathbf{r}) = \mathbf{0} \end{aligned} \tag{42}$$

This is due to the fact that $\vec{\omega}$ is a constant vector and

$$\nabla \times (x\mathbf{i} + \mathbf{yj} + \mathbf{zk}) = \mathbf{0}$$

.

In addition,

$$\nabla \times \mathbf{v} = \text{curl } \mathbf{v} = \nabla \times (\vec{\omega} \times \mathbf{r})$$

$$= \vec{\omega} (\nabla \bullet \mathbf{r}) - (\vec{\omega} \bullet \nabla) \mathbf{r} = 2\vec{\omega} \tag{43}$$

Notice that $\nabla \bullet \mathbf{r} = 3$ and $(\vec{\omega} \bullet \nabla) \mathbf{r} = \vec{\omega}$.

Thus, if a fluid element experiences pure rotation

$$\nabla \bullet \mathbf{v} = \mathbf{0} \tag{44}$$

$$\nabla \times \mathbf{v} = 2\vec{\omega} \tag{45}$$

If a fluid is irrotational and incompressible without sources and sinks,

$$\nabla \bullet \mathbf{v} = \mathbf{0} \tag{46}$$

$$\nabla \times \mathbf{v} = \mathbf{0} \tag{47}$$

Since $\nabla \times \mathbf{v} = \mathbf{0}$, it implies that the existence of a potential ϕ such that

$$\nabla \phi = \mathbf{v} \tag{48}$$

Thus,

$$\nabla \bullet \mathbf{v} = \nabla \bullet \nabla \phi = \nabla^{\mathbf{2}} \phi = \mathbf{0} \tag{49}$$

This is the so-called Laplace Equation! Generally speaking, in any continuously differentiable vector field $\mathbf{F}$ with zero divergence and curl in a simple region, the vector is the gradient of a solution of the Laplace equation.